

Crowd Behavior Dynamics: Entropic Path–Integral Model

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Abstract

We propose an entropic geometrical model of crowd behavior dynamics (with dissipative crowd kinematics), using Feynman action–amplitude formalism that operates on three synergetic levels: macro, meso and micro. The intent is to explain the dynamics of crowds simultaneously and consistently across these three levels, in order to characterize their geometrical properties particularly with respect to behavior regimes and the state changes between them. Its most natural statistical descriptor (order parameter) is crowd entropy S that satisfies the Prigogine’s extended second law of thermodynamics, $\partial_t S \geq 0$ (for any nonisolated multi-component system). Qualitative similarities and superpositions between individual and crowd configuration manifolds motivate our claim that *goal-directed crowd movement* operates under entropy conservation, $\partial_t S = 0$, while *naturally chaotic crowd dynamics* operates under (monotonically) increasing entropy function, $\partial_t S > 0$. Between these two distinct topological phases lies a phase transition with a chaotic inter-phase. Both inertial crowd dynamics and its dissipative kinematics represent diffusion processes on the crowd manifold governed by the Ricci flow.

Keywords: Crowd behavior dynamics, action–amplitude formalism, entropic crowd manifold, crowd turbulence, Ricci flow, topological phase transitions.

1 Introduction

Today it is well known that disembodied cognition is a myth, albeit one that has had profound influence in Western science since Rene Descartes and others gave it credence during the Scientific Revolution. In fact, the mind-body separation had much more to do with explanation of method than with explanation of the mind and cognition, yet it is with respect to the latter that its impact is most widely felt. We find it to be an unsustainable assumption in the realm of crowd behavior. Mental intention is (almost immediately) followed by a physical action, that is, a human or animal movement [68]. In animals, this physical action would be jumping, running, flying, swimming, biting or grabbing. In humans, it can be talking, walking, driving, or kicking, etc. Mathematical description of human/animal movement in terms of the corresponding neuro-musculo-skeletal equations of motion, for the purpose of prediction and control, is formulated within the realm of biodynamics (see [35, 47, 36, 37, 38, 39, 40, 41]).

The crowd (or, collective) behavior dynamics is clearly formed by some kind of *superposition*, *contagion*, *emergence*, or *convergence* from the individual agents’ behavior. According to the emergence theory [72], crowds begin as collectivities composed of people with mixed interests and

motives; especially in the case of less stable crowds (expressive, acting and protest crowds) norms may be vague and changing; people in crowds make their own rules as they go along. According to currently popular convergence theory (see [28, 59]), crowd behavior is not a product of the crowd itself, but is carried into the crowd by particular individuals, thus crowds amount to a convergence of likeminded individuals.

We propose that the contagion and convergence theories may be unified by acknowledging that both factors may coexist, even within a single scenario: we propose to refer to this third approach as *behavioral composition*. It represents a substantial shift from traditional analytical approaches, which have assumed either reduction of a whole into parts or the emergence of the whole from the parts. In particular, both contagion and convergence are related to social entropy, which is the natural decay of structure (such as law, organization, and convention) in a social system [9].

In this paper we attempt to formulate a geometrically predictive model-theory of crowd behavior dynamics, based on the previously formulated individual Life Space Foam concept [46] (see Appendix for the brief summary).

It is today well known that massive crowd movements can be precisely observed/monitored from satellites and all that one can see is crowd physics. Therefore, all involved psychology of individual crowd agents: cognitive, motivational and emotional – is only a non-transparent input (a hidden initial switch) for the fully observable crowd physics. In this paper we will label this initial switch as ‘mental preparation’ or ‘loading’, while the manifested physical action is labeled ‘execution’. We propose the entropy formulation of crowd dynamics as a three-step process involving individual behavior dynamics and collective behavior dynamics. The chaotic behavior phase-transitions embedded in crowd dynamics may give a formal description for a phenomenon called *crowd turbulence* by D. Helbing, depicting crowd disasters caused by the panic stampede that can occur at high pedestrian densities and which is a serious concern during mass events like soccer championship games or annual pilgrimage in Makkah (see [29, 30, 31, 54]).

2 Generic three-step crowd behavior dynamics

In this section we propose a generic crowd behavior dynamics as a three-step process based on a general partition function formalism. Note that the number of variables X_i in the standard *partition function* from statistical mechanics (see, e.g. [57]) need not be countable, in which case the set of coordinates $\{x^i\}$ becomes a field $\phi = \phi(x)$. The sum is replaced by the *Euclidean path integral* (that is a Wick-rotated Feynman transition amplitude in imaginary time, see subsection 3.4), as

$$Z(\phi) = \int \mathcal{D}[\phi] \exp[-H(\phi)].$$

More generally, in quantum field theory, instead of the field Hamiltonian $H(\phi)$ we have the action $S(\phi)$ of the theory. Both Euclidean path integral,

$$Z(\phi) = \int \mathcal{D}[\phi] \exp[-S(\phi)], \quad \text{real path integral in imaginary time} \quad (1)$$

and Lorentzian one,

$$Z(\phi) = \int \mathcal{D}[\phi] \exp[iS(\phi)], \quad \text{complex path integral in real time} \quad (2)$$

– represent quantum field theory (QFT) partition functions. We will give formal definitions of the above path integrals (i.e., general partition functions) in section 3. For the moment, we only remark that the Lorentzian path integral (2) gives a QFT generalization of the (nonlinear) Schrödinger equation, while the Euclidean path integral (1) in the (rectified) real time represents a statistical field theory (SFT) generalization of the Fokker–Planck equation.

Now, following the framework of the Prigogine’s Extended Second Law of Thermodynamics [60], $\partial_t S \geq 0$, for entropy S in any complex system described by its partition function, we formulate a generic crowd behavior dynamics, based on above partition functions, as the following three-step process:

1. Individual behavior dynamics (\mathcal{ID}) is a transition process from an entropy-growing “loading” phase of mental preparation, to the entropy-conserving “execution” phase of physical action. Formally, \mathcal{ID} is given by the phase-transition map:

$$\mathcal{ID} : \overbrace{\text{MENTAL PREPARATION}}^{\text{“LOADING”}: \partial_t S > 0} \Rightarrow \overbrace{\text{PHYSICAL ACTION}}^{\text{“EXECUTION”}: \partial_t S = 0} \quad (3)$$

defined by the individual (chaotic) phase-transition amplitude

$$\left\langle \text{PHYS. ACTION} \left| \text{CHAOS} \right| \text{MENTAL PREP.} \right\rangle_{\text{ID}}^{\partial_t S = 0} := \int \mathcal{D}[\Phi] e^{iS_{\text{ID}}[\Phi]},$$

where the right-hand-side is the Lorentzian path-integral (or complex path-integral in real time), with the individual behavior action

$$S_{\text{ID}}[\Phi] = \int_{t_{\text{ini}}}^{t_{\text{fin}}} L_{\text{ID}}[\Phi] dt,$$

where $L_{\text{ID}}[\Phi]$ is the *behavior Lagrangian*, consisting of *mental cognitive potential* and *physical kinetic energy*.

2. Aggregate behavior dynamics (\mathcal{AD}) represents the behavioral composition–transition map:

$$\mathcal{AD} : \sum_{i \in \text{AD}} \overbrace{\text{MENTAL PREPARATION}}^{\text{“LOADING”}: \partial_t S > 0} \Rightarrow \sum_{i \in \text{AD}} \overbrace{\text{PHYSICAL ACTION}_i}^{\text{“EXECUTION”}: \partial_t S = 0} \quad (4)$$

where the (weighted) aggregate sum is taken over all individual agents, assuming equipartition of the total behavioral energy. It is defined by the aggregate (chaotic) phase-transition amplitude

$$\left\langle \text{PHYS. ACTION} \left| \text{CHAOS} \right| \text{MENTAL PREP.} \right\rangle_{\text{AD}}^{\partial_t S = 0} := \int \mathcal{D}[\Phi] e^{-S_{\text{AD}}[\Phi]},$$

with the Euclidean path-integral in real time, that is the SFT–partition function, based on the aggregate behavior action

$$S_{\text{AD}}[\Phi] = \int_{t_{\text{ini}}}^{t_{\text{fin}}} L_{\text{AD}}[\Phi] dt, \quad \text{with} \quad L_{\text{AD}}[\Phi] = \sum_{i \in \text{AD}} L_{\text{ID}}^i[\Phi].$$

3. Crowd behavior dynamics (\mathcal{CD}) represents the cumulative transition map:

$$\mathcal{CD} : \sum_{i \in \mathcal{CD}} \overbrace{\text{MENTAL PREPARATION}}^{\text{"LOADING": } \partial_t S > 0} \Rightarrow \sum_{i \in \mathcal{CD}} \overbrace{\text{PHYSICAL ACTION}_i}^{\text{"EXECUTION": } \partial_t S = 0} \quad (5)$$

where the (weighted) cumulative sum is taken over all individual agents, assuming equipartition of the total behavior energy. It is defined by the crowd (chaotic) phase-transition amplitude

$$\left\langle \text{PHYS. ACTION} \left| \text{CHAOS} \right| \text{MENTAL PREP.} \right\rangle_{\mathcal{CD}} := \int \mathcal{D}[\Phi] e^{iS_{\mathcal{CD}}[\Phi]},$$

with the general Lorentzian path-integral, that is, the QFT-partition function), based on the crowd behavior action

$$S_{\mathcal{CD}}[\Phi] = \int_{t_{ini}}^{t_{fin}} L_{\mathcal{CD}}[\Phi] dt, \quad \text{with} \quad L_{\mathcal{CD}}[\Phi] = \sum_{i \in \mathcal{CD}} L_{\text{ID}}^i[\Phi] = \sum_{k=\# \text{ of ADs in CD}} L_{\text{AD}}^k[\Phi].$$

All three entropic phase-transition maps, \mathcal{ID} , \mathcal{AD} and \mathcal{CD} , are spatio-temporal biodynamic cognition systems [43], evolving within their respective configuration manifolds (i.e., sets of their respective degrees-of-freedom with equipartition of energy), according to biphasic action-functional formalisms with behavior-Lagrangian functions L_{ID} , L_{AD} and $L_{\mathcal{CD}}$, each consisting of:

1. Cognitive mental potential (which is a mental preparation for the physical action), and
2. Physical kinetic energy (which describes the physical action itself).

To develop \mathcal{ID} , \mathcal{AD} and \mathcal{CD} formalisms, we extend into a physical (or, more precisely, biodynamic) crowd domain a purely-mental individual Life-Space Foam (LSF) framework for motivational cognition [46], based on the quantum-probability concept.¹

¹The quantum probability concept is based on the following physical facts [50, 51]

1. The time-dependent Schrödinger equation represents a complex-valued generalization of the real-valued Fokker-Planck equation for describing the spatio-temporal probability density function for the system exhibiting continuous-time Markov stochastic process.
2. The Feynman path integral (including integration over continuous spectrum and summation over discrete spectrum) is a generalization of the time-dependent Schrödinger equation, including both continuous-time and discrete-time Markov stochastic processes.
3. Both Schrödinger equation and path integral give ‘physical description’ of any system they are modelling in terms of its physical energy, instead of an abstract probabilistic description of the Fokker-Planck equation.

Therefore, the Feynman path integral, as a generalization of the (nonlinear) time-dependent Schrödinger equation, gives a unique physical description for the general Markov stochastic process, in terms of the physically based generalized probability density functions, valid both for continuous-time and discrete-time Markov systems. Its basic consequence is this: a different way for calculating probabilities. The difference is rooted in the fact that *sum of squares is different from the square of sums*, as is explained in the following text. Namely, in Dirac-Feynman quantum formalism, each possible route from the initial system state A to the final system state B is called a history. This history comprises any kind of a route, ranging from continuous and smooth deterministic (mechanical-like) paths to completely discontinues and random Markov chains (see, e.g., [15]). Each history (labelled by index i) is quantitatively described by a complex number.

In this way, the overall probability of the system’s transition from some initial state A to some final state B is given not by adding up the probabilities for each history-route, but by ‘head-to-tail’ adding up the sequence of

The behavioral approach to ID , AD and CD is based on *entropic motor control* [33, 34], which deals with neuro-physiological feedback information and environmental uncertainty. The probabilistic nature of human motor action can be characterized by entropies at the level of the organism, task, and environment. Systematic changes in motor adaptation are characterized as task-organism and environment-organism tradeoffs in entropy. Such compensatory adaptations lead to a view of goal-directed motor control as the product of an underlying conservation of entropy across the task-organism-environment system. In particular, an experiment conducted in [34] examined the changes in entropy of the coordination of isometric force output under different levels of task demands and feedback from the environment. The goal of the study was to examine the hypothesis that human motor adaptation can be characterized as a process of entropy conservation that is reflected in the compensation of entropy between the task, organism motor output, and environment. Information entropy of the coordination dynamics relative phase of the motor output was made conditional on the idealized situation of human movement, for which the goal was always achieved. Conditional entropy of the motor output decreased as the error tolerance and feedback frequency were decreased. Thus, as the likelihood of meeting the task demands was decreased increased task entropy and/or the amount of information from the environment is reduced increased environmental entropy, the subjects of this experiment employed fewer coordination patterns in the force output to achieve the goal. The conservation of entropy supports the view that context dependent adaptations in human goal-directed action are guided fundamentally by natural law and provides a novel means of examining human motor behavior. This is fundamentally related to the *Heisenberg uncertainty principle* [51] and further supports the argument for the primacy of a probabilistic approach toward the study of biodynamic cognition systems.²

amplitudes making-up each route first (i.e., performing the sum-over-histories) – to get the total amplitude as a ‘resultant vector’, and then squaring the total amplitude to get the overall transition probability.

Here we emphasize that the domain of validity of the ‘quantum’ is not restricted to the microscopic world [73]. There are macroscopic features of classically behaving systems, which cannot be explained without recourse to the quantum dynamics. This field theoretic model leads to the view of the phase transition as a condensation that is comparable to the formation of fog and rain drops from water vapor, and that might serve to model both the gamma and beta phase transitions. According to such a model, the production of activity with long-range correlation in the brain takes place through the mechanism of spontaneous breakdown of symmetry (SBS), which has for decades been shown to describe long-range correlation in condensed matter physics. The adoption of such a field theoretic approach enables modelling of the whole cerebral hemisphere and its hierarchy of components down to the atomic level as a fully integrated macroscopic quantum system, namely as a macroscopic system which is a quantum system not in the trivial sense that it is made, like all existing matter, by quantum components such as atoms and molecules, but in the sense that some of its macroscopic properties can best be described with recourse to quantum dynamics (see [14] and references therein). Also, according to Freeman and Vitiello, *many-body quantum field theory* appears to be the only existing theoretical tool capable to explain the dynamic origin of long-range correlations, their rapid and efficient formation and dissolution, their interim stability in ground states, the multiplicity of coexisting and possibly non-interfering ground states, their degree of ordering, and their rich textures relating to sensory and motor facets of behaviors. It is historical fact that many-body quantum field theory has been devised and constructed in past decades exactly to understand features like ordered pattern formation and phase transitions in condensed matter physics that could not be understood in classical physics, similar to those in the brain.

²Our entropic action-amplitude formalism represents a kind of a generalization of the Haken-Kelso-Bunz (HKB) model of self-organization in the individual’s motor system [16, 55], including: multi-stability, phase transitions and hysteresis effects, presenting a contrary view to the purely feedback driven systems. HKB uses the concepts of synergetics (order parameters, control parameters, instability, etc) and the mathematical tools of nonlinearly coupled (nonlinear) dynamical systems to account for self-organized behavior both at the cooperative, coordinative level and at the level of the individual coordinating elements. The HKB model stands as a building block upon which numerous extensions and elaborations have been constructed. In particular, it has been possible to derive it from a realistic model of the cortical sheet in which neural areas undergo a reorganization that is mediated by intra- and inter-cortical connections. Also, the HKB model describes phase transitions (‘switches’) in coordinated

Yet, it is well known that humans possess more degrees of freedom than are needed to perform any defined motor task, but are required to co-ordinate them in order to reliably accomplish high-level goals, while faced with intense motor variability. In an attempt to explain how this takes place, Todorov and Jordan have formulated an alternative theory of human motor co-ordination based on the concept of stochastic optimal feedback control [70]. They were able to conciliate the requirement of goal achievement (e.g., grasping an object) with that of motor variability (biomechanical degrees of freedom). Moreover, their theory accommodates the idea that the human motor control mechanism uses internal ‘functional synergies’ to regulate task-irrelevant (redundant) movement.

Also, a developing field in coordination dynamics involves the theory of social coordination, which attempts to relate the DC to normal human development of complex social cues following certain patterns of interaction. This work is aimed at understanding how human social interaction is mediated by meta-stability of neural networks [49]. fMRI and EEG are particularly useful in mapping thalamocortical response to social cues in experimental studies. In particular, a new theory called the *Phi complex* has been developed by S. Kelso and collaborators, to provide experimental results for the theory of social coordination dynamics (see the recent nonlinear dynamics paper discussing social coordination and EEG dynamics [71]). According to this theory, a pair of phi rhythms, likely generated in the mirror neuron system, is the hallmark of human social coordination. Using a dual-EEG recording system, the authors monitored the interactions of eight pairs of subjects as they moved their fingers with and without a view of the other individual in the pair.

3 Formal model of crowd dynamics

In this section we formally develop a three-step crowd behavior dynamics, conceptualized by transition maps (3)–(4)–(5), in agreement with Haken’s synergetics [17, 18]. We first develop a macro-level individual behavior dynamics \mathcal{ID} . Then we generalize \mathcal{ID} into an ‘orchestrated’ behavioral-compositional crowd dynamics \mathcal{CD} , using a quantum-like micro-level formalism with individual agents representing ‘crowd quanta’. Finally we develop a meso-level aggregate statistical-field dynamics \mathcal{AD} , such that composition of the aggregates \mathcal{AD} makes-up the crowd.

3.1 Individual behavior dynamics (\mathcal{ID})

\mathcal{ID} transition map (3) is developed using the following action-amplitude formalism (see [46, 45]):

1. Macroscopically, as a smooth Riemannian n -manifold $M_{\mathcal{ID}}$ with steady force-fields and

human movement as follows: (i) when the agent begins in the anti-phase mode and speed of movement is increased, a spontaneous switch to symmetrical, in-phase movement occurs; (ii) this transition happens swiftly at a certain critical frequency; (iii) after the switch has occurred and the movement rate is now decreased the subject remains in the symmetrical mode, i.e. she does not switch back; and (iv) no such transitions occur if the subject begins with symmetrical, in-phase movements. The HKB dynamics of the order parameter relative phase as is given by a nonlinear first-order ODE:

$$\dot{\phi} = (\alpha + 2\beta r^2) \sin \phi - \beta r^2 \sin 2\phi,$$

where ϕ is the phase relation (that characterizes the observed patterns of behavior, changes abruptly at the transition and is only weakly dependent on parameters outside the phase transition), r is the oscillator amplitude, while α, β are coupling parameters (from which the critical frequency where the phase transition occurs can be calculated).

behavioral paths, modelled by a real-valued classical action functional $S_{\text{ID}}[\Phi]$, of the form

$$S_{\text{ID}}[\Phi] = \int_{t_{\text{ini}}}^{t_{\text{fin}}} L_{\text{ID}}[\Phi] dt,$$

(where macroscopic paths, fields and geometries are commonly denoted by an abstract field symbol Φ^i) with the potential-energy based Lagrangian L given by

$$L_{\text{ID}}[\Phi] = \int d^n x \mathcal{L}_{\text{ID}}(\Phi_i, \partial_{x^j} \Phi^i),$$

where \mathcal{L} is Lagrangian density, the integral is taken over all n local coordinates $x^j = x^j(t)$ of the ID, and $\partial_{x^j} \Phi^i$ are time and space partial derivatives of the Φ^i -variables over coordinates. The standard least action principle

$$\delta S_{\text{ID}}[\Phi] = 0,$$

gives, in the form of the Euler-Lagrangian equations, a shortest path, an extreme force-field, with a geometry of minimal curvature and topology without holes. We will see below that high Riemannian curvature generates chaotic behavior, while holes in the manifold produce topologically induced phase transitions.

2. Microscopically, as a collection of wildly fluctuating and jumping paths (histories), force-fields and geometries/topologies, modelled by a complex-valued adaptive path integral, formulated by defining a multi-phase and multi-path (multi-field and multi-geometry) transition amplitude from the entropy-growing state of Mental Preparation to the entropy-conserving state of Physical Action,

$$\langle \text{Physical Action} | \text{Mental Preparation} \rangle_{\text{ID}} := \int_{\text{ID}} \mathcal{D}[\Phi] e^{i S_{\text{ID}}[\Phi]} \quad (6)$$

where the functional ID-measure $\mathcal{D}[w\Phi]$ is defined as a weighted product

$$\mathcal{D}[w\Phi] = \lim_{N \rightarrow \infty} \prod_{s=1}^N w_s d\Phi_s^i, \quad (i = 1, \dots, n = \text{con} + \text{dis}), \quad (7)$$

representing an ∞ -dimensional neural network [46], with weights w_s updating by the general rule

$$\text{new value}(t+1) = \text{old value}(t) + \text{innovation}(t).$$

More precisely, the weights $w_s = w_s(t)$ in (54) are updated according to one of the two standard neural learning schemes, in which the micro-time level is traversed in discrete steps, i.e., if $t = t_0, t_1, \dots, t_s$ then $t+1 = t_1, t_2, \dots, t_{s+1}$:³

³The traditional neural networks approaches are known for their classes of functions they can represent. Here we are talking about functions in an *extensional* rather than merely *intensional* sense; that is, function can be read as input/output behavior [3, 4, 11, 26]. This limitation has been attributed to their low-dimensionality (the largest neural networks are limited to the order of 10^5 dimensions [53]). The proposed path integral approach represents a new family of function-representation methods, which potentially offers a basis for a fundamentally more expansive solution.

(a) A *self-organized, unsupervised* (e.g., Hebbian-like [27]) learning rule:

$$w_s(t+1) = w_s(t) + \frac{\sigma}{\eta}(w_s^d(t) - w_s^a(t)), \quad (8)$$

where $\sigma = \sigma(t)$, $\eta = \eta(t)$ denote *signal* and *noise*, respectively, while superscripts d and a denote *desired* and *achieved* micro-states, respectively; or

(b) A certain form of a *supervised gradient descent learning*:

$$w_s(t+1) = w_s(t) - \eta \nabla J(t), \quad (9)$$

where η is a small constant, called the *step size*, or the *learning rate*, and $\nabla J(n)$ denotes the gradient of the ‘performance hyper-surface’ at the t -th iteration.

(Note that we could also use a reward-based, reinforcement learning rule [69], in which system learns its optimal policy: $innovation(t) = |reward(t) - penalty(t)|$.)

In this way, we effectively derive a unique and globally smooth, causal and entropic phase-transition map (3), performed at a macroscopic (global) time-level from some initial time t_{ini} to the final time t_{fin} . Thus, we have obtained macro-objects in the ID: a single path described by Newtonian-like equation of motion, a single force-field described by Maxwellian-like field equations, and a single obstacle-free Riemannian geometry (with global topology without holes).

In particular, on the macro-level, we have the ID-paths, that is biodynamical trajectories generated by the Hamilton action principle

$$\delta S_{ID}[x] = 0,$$

with the Newtonian action $S_{ID}[x]$ given by (Einstein’s summation convention over repeated indices is always assumed)

$$S_{ID}[x] = \int_{t_{ini}}^{t_{fin}} [\varphi + \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j] dt, \quad (10)$$

where $\varphi = \varphi(t, x^i)$ denotes the mental LSF-potential field, while the second term,

$$T = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j,$$

represents the physical (biodynamic) kinetic energy generated by the Riemannian inertial metric tensor g_{ij} of the configuration biodynamic manifold M_{ID} (see Figure 1). The corresponding Euler-Lagrangian equations give the Newtonian equations of human movement

$$\frac{d}{dt} T_{\dot{x}^i} - T_{x^i} = F_i, \quad (11)$$

where subscripts denote the partial derivatives and we have defined the covariant muscular forces $F_i = F_i(t, x^i, \dot{x}^i)$ as negative gradients of the mental potential $\varphi(x^i)$,

$$F_i = -\varphi_{x^i}. \quad (12)$$

Equation (11) can be put into the standard Lagrangian form as

$$\frac{d}{dt} L_{\dot{x}^i} = L_{x^i}, \quad \text{with} \quad L = T - \varphi(x^i), \quad (13)$$

or (using the Legendre transform) into the forced, dissipative Hamiltonian form [36, 39]

$$\dot{x}^i = \partial_{p_i} H + \partial_{p_i} R, \quad \dot{p}_i = F_i - \partial_{x^i} H + \partial_{x^i} R, \quad (14)$$

where p_i are the generalized momenta (canonically-conjugate to the coordinates x^i), $H = H(p, x)$ is the Hamiltonian (total energy function) and $R = R(p, x)$ is the general dissipative function.

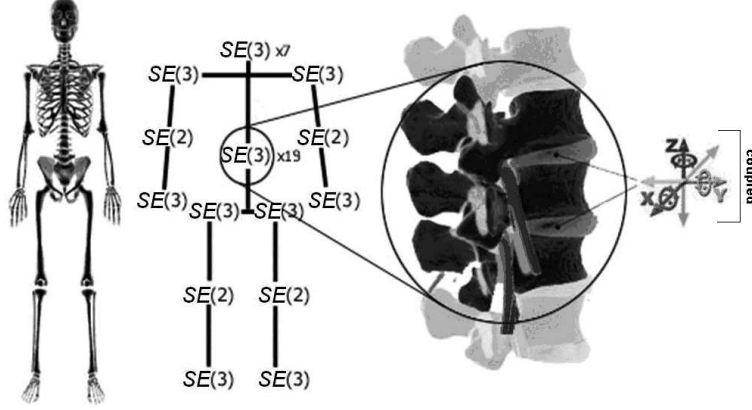


Figure 1: Riemannian configuration manifold M_{ID} of human biodynamics is defined as a topological product $M = \prod_i SE(3)^i$ of constrained Euclidean $SE(3)$ -groups of rigid body motion in 3D Euclidean space (see [41, 44]), acting in all major (synovial) human joints. The manifold M is a dynamical structure activated/controlled by potential covariant forces (12) produced by a synergetic action of about 640 skeletal muscles [39].

The human motor system possesses many independently controllable components that often allow for more than a single movement pattern to be performed in order to achieve a goal. Hence, the motor system is endowed with a high level of adaptability to different tasks and also environmental contexts [34]. The multiple $SE(3)$ -dynamics applied to human musculo-skeletal system gives the fundamental law of biodynamics, which is the *covariant force law*:⁴

$$\text{Force co-vector field} = \text{Mass distribution} \times \text{Acceleration vector-field}, \quad (15)$$

which is formally written:

$$F_i = g_{ij} a^j, \quad (i, j = 1, \dots, n = \dim(M)),$$

where F_i are the covariant force/torque components. g_{ij} is the inertial metric tensor of the configuration Riemannian manifold $M = \prod_i SE(3)^i$ (g_{ij} defines the mass-distribution of the human body), while a^j are the contravariant components of the linear and angular acceleration vector-field. Both Lagrangian and (topologically equivalent) Hamiltonian development of the covariant

⁴This fundamental biodynamic law states that contrary to common perception, acceleration and force are not quantities of the same nature: while acceleration is a non-inertial vector-field, force is an inertial co-vector-field. This apparently insignificant difference becomes crucial in injury prediction/prevention, especially in its derivative form in which the ‘massless jerk’ ($= \dot{a}$) is relatively benign, while the ‘massive jolt’ ($= \dot{F}$) is deadly.

force law is fully elaborated in [39, 40, 41, 44]. This is consistent with the postulation that human action is guided primarily by natural law [56].

On the micro-ID level, instead of each single trajectory defined by the Newtonian equation of motion (11), we have an ensemble of fluctuating and crossing paths on the configuration manifold M with weighted probabilities (of the unit total sum). This ensemble of micro-paths is defined by the simplest instance of our adaptive path integral (6), similar to the Feynman's original sum over histories,

$$\langle \text{Physical Action} | \text{Mental Preparation} \rangle_M = \int_{\text{ID}} \mathcal{D}[wx] e^{iS[x]}, \quad (16)$$

where $\mathcal{D}[wx]$ is the functional ID-measure on the space of all weighted paths, and the exponential depends on the action $S_{\text{ID}}[x]$ given by (10).

3.2 Crowd behavioral-compositional dynamics (\mathcal{CD})

In this subsection we develop a generic crowd \mathcal{CD} , as a unique and globally smooth, causal and entropic phase-transition map (5), in which agents (or, crowd's individual entities) can be both humans and robots. This crowd behavior action takes place in a crowd smooth Riemannian $3n$ -manifold M . Recall from Figure 1 that each individual segment of a human body moves in the Euclidean 3-space \mathbb{R}^3 according to its own constrained $SE(3)$ -group. Similarly, each individual agent's trajectory, $x^i = x^i(t)$, $i = 1, \dots, n$, is governed by the Euclidean $SE(2)$ -group of rigid body motions in the plane. (Recall that a Lie group $SE(2) \equiv SO(2) \times \mathbb{R}$ is a set of all 3×3 -matrices of the form:

$$\begin{bmatrix} \cos \theta & \sin \theta & x \\ -\sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{bmatrix},$$

including both rigid translations (i.e., Cartesian x, y -coordinates) and rotation matrix $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

in Euclidean plane \mathbb{R}^2 (see [41, 44]).) The crowd configuration manifold M is defined as a union of Euclidean $SE(2)$ -groups for all n individual agents in the crowd, that is crowd's configuration $3n$ -manifold is defined as a set

$$\begin{aligned} M &= \sum_{k=1}^n SE(2)^k \equiv \sum_{k=1}^n SO(2)^k \times \mathbb{R}^k, \\ \text{coordinated by } \mathbf{x}^k &= \{x^k, y^k, \theta^k\}, \text{ (for } k = 1, 2, \dots, n). \end{aligned} \quad (17)$$

In other words, the crowd configuration manifold M is a *dynamical planar graph* with individual agents' $SE(2)$ -groups of motion in the vertices and time-dependent inter-agent distances $I_{ij} = [x^i(t_i) - x^j(t_j)]$ as edges.

Similarly to the individual case, the crowd action functional includes mental cognitive potential and physical kinetic energy, formally given by (with $i, j = 1, \dots, 3n$):

$$\begin{aligned} A[x^i, x^j; t_i, t_j] &= \frac{1}{2} \int_{t_i} \int_{t_j} \delta(I_{ij}^2) \dot{x}^i(t_i) \dot{x}^j(t_j) dt_i dt_j + \frac{1}{2} \int_t g_{ij} \dot{x}^i(t) \dot{x}^j(t) dt, \\ \text{with } I_{ij}^2 &= [x^i(t_i) - x^j(t_j)]^2, \quad \text{where } IN \leq t_i, t_j, t \leq OUT. \end{aligned} \quad (18)$$

The first term in (18) represents the mental potential for the interaction between any two agents x^i and x^j within the total crowd matrix x^{ij} . (Although, formally, this term contains cognitive velocities, it still represents ‘potential energy’ from the physical point of view.) It is defined as a double integral over a delta function of the square of interval I^2 between two points on the paths in their individual cognitive LSFs. Interaction occurs only when this LSF–distance between the two agents x^i and x^j vanishes. Note that the cognitive intentions of any two agents generally occur at different times t_i and t_j unless $t_i = t_j$, when cognitive synchronization occurs. This term effectively represents the *crowd cognitive controller* (see [45]).

The second term in (18) represents kinetic energy of the physical interaction of agents. Namely, after the above cognitive synchronization is completed, the second term of physical kinetic energy is activated in the common CD manifold, reducing it to just one of the agents’ individual manifolds, which is equivalent to the center-of-mass segment in the human musculo-skeletal system. Therefore, from (18) we can derive a generic Euler–Lagrangian dynamics that is a composition of (13), which also means that we have in place a generic Hamiltonian dynamics that is an amalgamate of (14), and the crowd covariant force law (15), the governing law of crowd biodynamics:

$$\begin{aligned} \text{Crowd force co-vector field} &= \text{Crowd mass distribution} \times \text{Crowd acceleration vector-field,} \\ \text{formally: } F_i &= g_{ij}a^j, \quad \text{where } g_{ij} \text{ is the inertial metric tensor of crowd manifold } M. \end{aligned} \quad (19)$$

The left-hand side of this equation defines forces acting on the crowd, while right-hand defines its mass distribution coupled to the crowd kinematics (\mathcal{CK} , described in the next subsection).

At the slave level, the adaptive path integral, representing an ∞ –dimensional neural network, corresponding to the crowd behavior action (18), reads

$$\langle \text{Physical Action} \mid \text{Mental Preparation} \rangle_{\text{CD}} = \int_{\text{CD}} \mathcal{D}[w, x, y] e^{iA[x, y; t_i, t_j]}, \quad (20)$$

where the Lebesgue-type integration is performed over all continuous paths $x^i = x^i(t_i)$ and $y^j = y^j(t_j)$, while summation is performed over all associated discrete Markov fluctuations and jumps. The symbolic differential in the path integral (20) represents an adaptive path measure, defined as the weighted product

$$\mathcal{D}[w, x, y] = \lim_{N \rightarrow \infty} \prod_{s=1}^N w_{ij}^s dx^i dy^j, \quad (i, j = 1, \dots, n). \quad (21)$$

The quantum–field path integral (20)–(21) defines the microstate \mathcal{CD} –level, an ensemble of fluctuating and crossing paths on the crowd $3n$ –manifold M .

3.3 Dissipative crowd kinematics (\mathcal{CK})

The crowd action (18) with its amalgamate Lagrangian dynamics (13) and amalgamate Hamiltonian dynamics (14), as well as the crowd force law (19) define the macroscopic crowd dynamics, \mathcal{CD} . Suppose, for a moment, that \mathcal{CD} is force–free and dissipation free, therefore conservative. Now, the basic characteristic of the conservative Lagrangian/Hamiltonian systems evolving in the phase space spanned by the system coordinates and their velocities/momenta, is that their *flow* φ_t^L (explained below) preserves the phase–space volume. This is proposed by the Liouville theorem, which is the well known fact in statistical mechanics. However, the preservation of the phase

volume causes structural instability of the conservative system, i.e., the phase-space spreading effect by which small phase regions R_t will tend to get distorted from the initial one R_o during the conservative system evolution. This problem, governed by entropy growth ($\partial_t S > 0$), is much more serious in higher dimensions than in lower dimensions, since there are so many ‘directions’ in which the region can locally spread (see [63, 41]). This phenomenon is related to *conservative Hamiltonian chaos* (see section 4 below).

However, this situation is not very frequent in case of ‘organized’ human crowd. Its self-organization mechanisms are clearly much stronger than the conservative statistical mechanics effects, which we interpret in terms of Prigogine’s dissipative structures. Formally, if dissipation of energy in a system is much stronger than its inertial characteristics, then instead of the second-order Newton–Lagrangian dynamic equations of motion, we are actually dealing with the first-order driftless (non-acceleration, non-inertial) kinematic equations of motion, which is related to *dissipative chaos* [61]. Briefly, the dissipative crowd flow can be depicted like this: from the set of initial conditions for individual agents, the crowd evolves in time towards the set of the corresponding *entangled attractors*,⁵ which are mutually separated by fractal (non-integer dimension) separatrices.

In this subsection we elaborate on the dissipative crowd kinematics (\mathcal{CK}), which is self-controlled and dominates the \mathcal{CD} if the crowd’s inertial forces are much weaker than the crowd’s dissipation of energy, presented here in the form of nonlinear velocity controllers.

Recall that the essential concept in dynamical systems theory is the notion of a *vector-field* (that we will denote by a boldface symbol), which assigns a tangent vector to each point p in the manifold in case. In particular, \mathbf{v} is a gradient vector-field if it equals the gradient of some scalar function. A *flow-line* of a vector-field \mathbf{v} is a path $\gamma(t)$ satisfying the vector ODE, $\dot{\gamma}(t) = \mathbf{v}(\gamma(t))$, that is, \mathbf{v} yields the velocity field of the path $\gamma(t)$. The set of all flow lines of a vector-field \mathbf{v} comprises its flow φ_t that is (technically, see e.g., [41, 44]) a one-parameter Lie group of diffeomorphisms

⁵Recall that quantum entanglement is a quantum mechanical phenomenon in which the quantum states of two or more objects are linked together so that one object can no longer be adequately described without full mention of its counterpart – even though the individual objects may be spatially separated. This interconnection leads to correlations between observable physical properties of remote systems. The related phenomenon of wave-function collapse gives an impression that measurements performed on one system instantaneously influence the other systems entangled with the measured system, even when far apart.

Entanglement has many applications in quantum information theory. Mixed state entanglement can be viewed as a resource for quantum communication. A common measure of entanglement is the entropy of a mixed quantum state (see, e.g. [51]). Since a mixed quantum state ρ is a probability distribution over a quantum ensemble, this leads naturally to the definition of the *von Neumann entropy*, $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$, which is obviously similar to the classical *Shannon entropy* for probability distributions (p_1, \dots, p_n) , defined as $S(p_1, \dots, p_n) = -\sum_i p_i \log_2 p_i$. As in statistical mechanics, one can say that the more uncertainty (number of microstates) the system should possess, the larger is its entropy. Entropy gives a tool which can be used to quantify entanglement. If the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems.

The most popular issue in a research on dissipative quantum brain modelling has been *quantum entanglement* between the *brain* and its *environment* [65, 66], where the brain–environment system has an entangled ‘memory’ state, identified with the ground (vacuum) state $|0\rangle_{\mathcal{N}}$, that cannot be factorized into two single-mode states. (In the Vitiello–Pessa dissipative quantum brain model [65, 66], the evolution of the \mathcal{N} -coded memory system was represented as a trajectory of given initial condition running over time-dependent states $|0(t)\rangle_{\mathcal{N}}$, each one minimizing the free energy functional.) Similar to this microscopic brain–environment entanglement, we propose a kind of *macroscopic entanglement* between the operating modes of the *crowd behavior controller* and its biodynamics, which can be considered as a ‘long-range correlation’.

Applied externally to the dimension of the crowd $3n$ –manifold M , entanglement effectively reduces the number of active degrees of freedom in (17).

(smooth bijective functions) generated by a vector-field \mathbf{v} on M , such that

$$\varphi_t \circ \varphi_s = \varphi_{t+s}, \quad \varphi_0 = \text{identity}, \quad \text{which gives: } \gamma(t) = \varphi_t(\gamma(0)).$$

Analytically, a vector-field \mathbf{v} is defined as a set of autonomous ODEs. Its solution gives the flow φ_t , consisting of integral curves (or, flow lines) $\gamma(t)$ of the vector-field, such that all the vectors from the vector-field are tangent to integral curves at different representative points $p \in M$. In this way, through every representative point $p \in M$ passes both a curve from the flow and its tangent vector from the vector-field. Geometrically, vector-field is defined as a cross-section of the tangent bundle TM of the manifold M .

In general, given an n D frame $\{\partial_i\} \equiv \{\partial/\partial x^i\}$ on a smooth n -manifold M (that is, a basis of tangent vectors in a local coordinate chart $x^i = (x^1, \dots, x^n) \subset M$), we can define any vector-field \mathbf{v} on M by its components $v^i = v^i(t)$ as

$$\mathbf{v} = v^i \partial_i = v^i \frac{\partial}{\partial x^i} = v^1 \frac{\partial}{\partial x^1} + \dots + v^n \frac{\partial}{\partial x^n}.$$

Thus, a vector-field $\mathbf{v} \in \mathcal{X}(M)$ (where $\mathcal{X}(M)$ is the set of all smooth vector-fields on M) is actually a differential operator that can be used to differentiate any smooth scalar function $f = f(x^1, \dots, x^n)$ on M , as a *directional derivative* of f in the direction of \mathbf{v} . This is denoted simply $\mathbf{v}f$, such that

$$\mathbf{v}f = v^i \partial_i f = v^i \frac{\partial f}{\partial x^i} = v^1 \frac{\partial f}{\partial x^1} + \dots + v^n \frac{\partial f}{\partial x^n}.$$

In particular, if $\mathbf{v} = \dot{\gamma}(t)$ is a velocity vector-field of a space curve $\gamma(t) = (x^1(t), \dots, x^n(t))$, defined by its components $v^i = \dot{x}^i(t)$, directional derivative of $f(x^i)$ in the direction of \mathbf{v} becomes

$$\mathbf{v}f = \dot{x}^i \partial_i f = \frac{dx^i}{dt} \frac{\partial f}{\partial x^i} = \frac{df}{dt} = \dot{f},$$

which is a rate-of-change of f along the curve $\gamma(t)$ at a point $x^i(t)$.

Given two vector-fields, $\mathbf{u} = u^i \partial_i$, $\mathbf{v} = v^i \partial_i \in \mathcal{X}(M)$, their Lie bracket (or, commutator) is another vector-field $[\mathbf{u}, \mathbf{v}] \in \mathcal{X}(M)$, defined by

$$[\mathbf{u}, \mathbf{v}] = \mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u} = u^i \partial_i v^j \partial_j - v^j \partial_j u^i \partial_i,$$

which, applied to any smooth function f on M , gives

$$[\mathbf{u}, \mathbf{v}](f) = \mathbf{u}(\mathbf{v}(f)) - \mathbf{v}(\mathbf{u}(f)).$$

The Lie bracket measures the failure of ‘mixed directional derivatives’ to commute. Clearly, mixed partial derivatives *do* commute, $[\partial_i, \partial_j] = 0$, while in general it is *not* the case, $[\mathbf{u}, \mathbf{v}] \neq 0$. In addition, suppose that \mathbf{u} generates the flow φ_t and \mathbf{v} generates the flow φ_s . Then, for any smooth function f on M , we have at any point p on M ,

$$[\mathbf{u}, \mathbf{v}](f)(p) = \frac{\partial^2}{\partial t \partial s} (f(\varphi_s(\varphi_t(p))) - f(\varphi_t(\varphi_s(p))),$$

which means that in $f(\varphi_s(\varphi_t(p)))$ we are starting at p , flowing along \mathbf{v} a little bit, then along \mathbf{u} a little bit, and then evaluating f , while in $f(\varphi_t(\varphi_s(p)))$ we are flowing first along \mathbf{u} and then \mathbf{v} . Therefore, the Lie bracket infinitesimally measures how these flows fail to commute.

The Lie bracket satisfies the following three properties (for any three vector-fields $\mathbf{u}, \mathbf{v}, \mathbf{w} \in M$ and two constants a, b – thus forming a Lie algebra on the crowd manifold M):

- (i) $[\mathbf{u}, \mathbf{v}] = -[\mathbf{v}, \mathbf{u}]$ – skew-symmetry;
- (ii) $[\mathbf{u}, a\mathbf{v} + b\mathbf{w}] = a[\mathbf{u}, \mathbf{v}] + b[\mathbf{u}, \mathbf{w}]$ – bilinearity; and
- (iii) $[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] = 0$ – Jacobi identity.

A new set of vector-fields on M can be generated by repeated Lie brackets of $\mathbf{u}, \mathbf{v}, \mathbf{w} \in M$.

The Lie bracket is a standard tool in geometric nonlinear control theory (see, e.g. [41, 44]). Its action on vector-fields can be best visualized using the popular car parking example, in which the driver has two different vector-field transformations at his disposal. They can turn the steering wheel, or they can drive the car forward or backward. Here, we specify the state of a car by four coordinates: the (x, y) coordinates of the center of the rear axle, the direction θ of the car, and the angle ϕ between the front wheels and the direction of the car. l is the constant length of the car. Therefore, the 4D configuration manifold of a car is a set $M \equiv SO(2) \times \mathbb{R}^2$, coordinated by $\mathbf{x} = \{x, y, \theta, \phi\}$, which is slightly more complicated than the individual crowd agent's 3D configuration manifold $SE(2) \equiv SO(2) \times \mathbb{R}$, coordinated by $\mathbf{x} = \{x, y, \theta\}$. The driftless car kinematics can be defined as a vector ODE:

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}) c_1 + \mathbf{v}(\mathbf{x}) c_2, \quad (22)$$

with two vector-fields, $\mathbf{u}, \mathbf{v} \in \mathcal{X}(M)$, and two scalar control inputs, c_1 and c_2 . The infinitesimal car-parking transformations will be the following vector-fields

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &\equiv \text{DRIVE} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} + \frac{\tan \phi}{l} \frac{\partial}{\partial \theta} \equiv \begin{pmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{l} \tan \phi \\ 0 \end{pmatrix}, \\ \text{and } \mathbf{v}(\mathbf{x}) &\equiv \text{STEER} = \frac{\partial}{\partial \phi} \equiv \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \end{aligned}$$

The car kinematics (22) therefore expands into a matrix ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \text{DRIVE} \cdot c_1 + \text{STEER} \cdot c_2 \equiv \begin{pmatrix} \cos \theta \\ \sin \theta \\ \frac{1}{l} \tan \phi \\ 0 \end{pmatrix} \cdot c_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot c_2.$$

However, STEER and DRIVE do not commute (otherwise we could do all your steering at home before driving of on a trip). Their combination is given by the Lie bracket

$$[\mathbf{v}, \mathbf{u}] \equiv [\text{STEER}, \text{DRIVE}] = \frac{1}{l \cos^2 \phi} \frac{\partial}{\partial \theta} \equiv \text{WRIGGLE}.$$

The operation $[\mathbf{v}, \mathbf{u}] \equiv \text{WRIGGLE} \equiv [\text{STEER}, \text{DRIVE}]$ is the infinitesimal version of the sequence of transformations: steer, drive, steer back, and drive back, i.e.,

$$\{\text{STEER}, \text{DRIVE}, \text{STEER}^{-1}, \text{DRIVE}^{-1}\}.$$

Now, WRIGGLE can get us out of some parking spaces, but not tight ones: we may not have enough room to WRIGGLE out. The usual tight parking space restricts the DRIVE transformation, but not STEER. A truly tight parking space restricts STEER as well by putting your front wheels against the curb.

Fortunately, there is still another commutator available:

$$\begin{aligned} [\mathbf{u}, [\mathbf{v}, \mathbf{u}]] &\equiv [\text{DRIVE}, [\text{STEER}, \text{DRIVE}]] = [[\mathbf{u}, \mathbf{v}], \mathbf{u}] \equiv \\ [\text{DRIVE}, \text{WRIGGLE}] &= \frac{1}{l \cos^2 \phi} \left(\sin \theta \frac{\partial}{\partial x} - \cos \theta \frac{\partial}{\partial y} \right) \equiv \text{SLIDE}. \end{aligned}$$

The operation $[[\mathbf{u}, \mathbf{v}], \mathbf{u}] \equiv \text{SLIDE} \equiv [\text{DRIVE}, \text{WRIGGLE}]$ is a displacement at right angles to the car, and can get us out of any parking place. We just need to remember to steer, drive, steer back, drive some more, steer, drive back, steer back, and drive back:

$$\{\text{STEER}, \text{DRIVE}, \text{STEER}^{-1}, \text{DRIVE}, \text{STEER}, \text{DRIVE}^{-1}, \text{STEER}^{-1}, \text{DRIVE}^{-1}\}.$$

We have to reverse steer in the middle of the parking place. This is not intuitive, and no doubt is part of a common problem with parallel parking.

Thus, from only two controls, c_1 and c_2 , we can form the vector-fields $\text{DRIVE} \equiv \mathbf{u}$, $\text{STEER} \equiv \mathbf{v}$, $\text{WRIGGLE} \equiv [\mathbf{v}, \mathbf{u}]$, and $\text{SLIDE} \equiv [[\mathbf{u}, \mathbf{v}], \mathbf{u}]$, allowing us to move anywhere in the car configuration manifold $M \equiv SO(2) \times \mathbb{R}^2$. All above computations are straightforward in *Mathematica*^{TM6} if we define the following three symbolic functions:

1. Jacobian matrix: $\text{JacMat}[\mathbf{v_List}, \mathbf{x_List}] := \text{Outer}[\mathbf{D}, \mathbf{v}, \mathbf{x}];$
2. Lie bracket: $\text{LieBrc}[\mathbf{u_List}, \mathbf{v_List}, \mathbf{x_List}] := \text{JacMat}[\mathbf{v}, \mathbf{x}] \cdot \mathbf{u} - \text{JacMat}[\mathbf{u}, \mathbf{x}] \cdot \mathbf{v};$
3. Repeated Lie bracket: $\text{Adj}[\mathbf{u_List}, \mathbf{v_List}, \mathbf{x_List}, \mathbf{k_}] :=$
 $\text{If}[\mathbf{k} == 0, \mathbf{v}, \text{LieBrc}[\mathbf{u}, \text{Adj}[\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{k} - 1], \mathbf{x}]];$

In case of the human crowd, we have a slightly simpler, but multiplied problem, i.e., superposition of n individual agents' motions. So, we can define the dissipative crowd kinematics as a system of n vector ODEs:

$$\dot{\mathbf{x}}^k = \mathbf{u}^k(\mathbf{x}) c_1^k + \mathbf{v}^k(\mathbf{x}) c_2^k, \quad \text{where} \quad (23)$$

$$\begin{aligned} \mathbf{u}^k(\mathbf{x}) &\equiv \text{DRIVE}^k = \cos^k \theta \frac{\partial}{\partial x^k} + \sin^k \theta \frac{\partial}{\partial y^k} \equiv \begin{pmatrix} \cos^k \theta \\ \sin^k \theta \\ 0 \end{pmatrix}, \quad \text{and} \\ \mathbf{v}^k(\mathbf{x}) &\equiv \text{STEER}^k = \frac{\partial}{\partial \theta^k} \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{while } c_1^k \text{ and } c_2^k \text{ are crowd controls.} \end{aligned}$$

Thus, the crowd kinematics (23) expands into the matrix ODE:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \text{DRIVE}^k \cdot c_1^k + \text{STEER}^k \cdot c_2^k \equiv \begin{pmatrix} \cos^k \theta \\ \sin^k \theta \\ 0 \end{pmatrix} \cdot c_1^k + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \cdot c_2^k. \quad (24)$$

A 3D simulation of random, dissipative crowd kinematics (23)–(24) of 120 penguin-like $SE(2)$ –robots, developed in C++/DirX is presented in Figure 2.

⁶The above computations could instead be done in other available packages, such as Maple, by suitably translating the provided example code.



Figure 2: Driving and steering random $SE(2)$ -dynamics of 120 penguin-like robots (with embedded collision-detection). Compare with [2].

The dissipative crowd kinematics (23)–(24) obeys the set of n -tuple integral rules of motion that are similar (though slightly simpler) to the above rules of the car kinematics, including the following derived vector-fields:

$$\text{WRIGGLE}^k \equiv [\text{STEER}^k, \text{DRIVE}^k] \equiv [\mathbf{v}^k, \mathbf{u}^k] \quad \text{and} \quad \text{SLIDE}^k \equiv [\text{DRIVE}^k, \text{WRIGGLE}^k] \equiv [[\mathbf{u}^k, \mathbf{v}^k], \mathbf{u}^k].$$

Thus, controlled by the two vector controls c_1^k and c_2^k , the crowd can form the vector-fields: $\text{DRIVE} \equiv \mathbf{u}^k$, $\text{STEER} \equiv \mathbf{v}^k$, $\text{WRIGGLE} \equiv [\mathbf{v}^k, \mathbf{u}^k]$, and $\text{SLIDE} \equiv [[\mathbf{u}^k, \mathbf{v}^k], \mathbf{u}^k]$, allowing it to move anywhere within its configuration manifold M given by (17). Solution of the dissipative crowd kinematics (23)–(24) defines the dissipative crowd flow, ϕ_t^K .

Now, the general $\mathcal{CD}\text{--}\mathcal{CK}$ crowd behavior can be defined as a amalgamate flow (behavior–Lagrangian flow, ϕ_t^L , plus dissipative kinematic flow, ϕ_t^K) on the crowd manifold M defined by (17),

$$C_t = \phi_t^L + \phi_t^K : t \mapsto (M(t), g(t)),$$

which is a one-parameter family of homeomorphic (topologically equivalent) Riemannian manifolds⁷ ($M, g = g_{ij}$), parameterized by a ‘time’ parameter t . That is, C_t can be used for describing

⁷Proper differentiation of vector and tensor fields on a smooth Riemannian manifold (like the crowd $3n$ –manifold

smooth deformations of the crowd manifold M over time. The manifold family $(M(t), g(t))$ at time t determines the manifold family $(M(t+dt), g(t+dt))$ at an infinitesimal time $t+dt$ into the future, according to some prescribed geometric flow, like the celebrated *Ricci flow* [22, 23, 25, 24] (that was an instrument for a proof of a 100-year old Poincaré conjecture),

$$\partial_t g_{ij}(t) = -2R_{ij}(t), \quad (25)$$

where R_{ij} is the Ricci curvature tensor of the crowd manifold M and $\partial_t g(t)$ is defined as

$$\partial_t g(t) \equiv \frac{d}{dt} g(t) := \lim_{dt \rightarrow 0} \frac{g(t+dt) - g(t)}{dt}. \quad (26)$$

3.4 Aggregate behavioral–compositional dynamics (\mathcal{AD})

To formally develop the meso-level aggregate behavioral–compositional dynamics (\mathcal{AD}), we start with the crowd path integral (20), which can be redefined if we Wick–rotate the time variable t to imaginary values, $t \mapsto \tau = it$, thereby transforming the Lorentzian path integral in real time into the Euclidean path integral in imaginary time. Furthermore, if we rectify the time axis back to

M) is performed using the *Levi–Civita covariant derivative* (see, e.g., [41, 44]). Formally, let M be a Riemannian N –manifold with the tangent bundle TM and a local coordinate system $\{x^i\}_{i=1}^N$ defined in an open set $U \subset M$. The covariant derivative operator, $\nabla_X : C^\infty(TM) \rightarrow C^\infty(TM)$, is the unique linear map such that for any vector-fields X, Y, Z , constant c , and scalar function f the following properties are valid:

$$\nabla_{X+cY} = \nabla_X + c\nabla_Y, \quad \nabla_X(Y+fZ) = \nabla_X Y + (Xf)Z + f\nabla_X Z, \quad \nabla_X Y - \nabla_Y X = [X, Y],$$

where $[X, Y]$ is the Lie bracket of X and Y . In local coordinates, the metric g is defined for any orthonormal basis $(\partial_i = \partial/\partial x^i)$ in $U \subset M$ by $g_{ij} = g(\partial_i, \partial_j) = \delta_{ij}$, $\partial_k g_{ij} = 0$. Then the affine *Levi–Civita connection* is defined on M by

$$\nabla_{\partial_i} \partial_j = \Gamma_{ij}^k \partial_k, \quad \text{where} \quad \Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) \quad \text{are the Christoffel symbols.}$$

Now, using the covariant derivative operator ∇_X we can define the *Riemann curvature* $(3, 1)$ –tensor \mathfrak{Rm} by

$$\mathfrak{Rm}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

which measures the curvature of the manifold by expressing how noncommutative covariant differentiation is. The $(3, 1)$ –components R_{ijk}^l of \mathfrak{Rm} are defined in $U \subset M$ by

$$\mathfrak{Rm}(\partial_i, \partial_j) \partial_k = R_{ijk}^l \partial_l, \quad \text{or} \quad R_{ijk}^l = \partial_i \Gamma_{jk}^l - \partial_j \Gamma_{ik}^l + \Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l.$$

Also, the Riemann $(4, 0)$ –tensor $R_{ijkl} = g_{lm} R_{ijk}^m$ is defined as the g –based inner product on M ,

$$R_{ijkl} = \langle \mathfrak{Rm}(\partial_i, \partial_j) \partial_k, \partial_l \rangle.$$

The first and second Bianchi identities for the Riemann $(4, 0)$ –tensor R_{ijkl} hold,

$$R_{ijkl} + R_{jkil} + R_{kijl} = 0, \quad \nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0,$$

while the twice contracted second Bianchi identity reads: $2\nabla_j R_{ij} = \nabla_i R$.

The $(0, 2)$ *Ricci tensor* \mathfrak{Rc} is the trace of the Riemann $(3, 1)$ –tensor \mathfrak{Rm} ,

$$\mathfrak{Rc}(Y, Z) + \text{tr}(X \rightarrow \mathfrak{Rm}(X, Y)Z), \quad \text{so that} \quad \mathfrak{Rc}(X, Y) = g(\mathfrak{Rm}(\partial_i, X) \partial_i, Y),$$

Its components $R_{jk} = \mathfrak{Rc}(\partial_j, \partial_k)$ are given in $U \subset M$ by the contraction

$$R_{jk} = R_{ijk}^i, \quad \text{or} \quad R_{jk} = \partial_i \Gamma_{jk}^i - \partial_k \Gamma_{ji}^i + \Gamma_{mi}^i \Gamma_{jk}^m - \Gamma_{mk}^i \Gamma_{ji}^m.$$

Finally, the scalar curvature R is the trace of the Ricci tensor \mathfrak{Rc} , given in $U \subset M$ by: $R = g^{ij} R_{ij}$.

the real line, we get the adaptive SFT-partition function as our proposed \mathcal{AD} -model:

$$\langle \text{Physical Action} | \text{Mental Preparation} \rangle_{\text{AD}} = \int_{\text{CD}} \mathcal{D}[w, x, y] e^{-A[x, y; t_i, t_j]}. \quad (27)$$

The adaptive \mathcal{AD} -transition amplitude $\langle \text{Physical Action} | \text{Mental Preparation} \rangle_{\text{AD}}$ as defined by the SFT-partition function (27) is a general model of a *Markov stochastic process*. Recall that Markov process is a random process characterized by a *lack of memory*, i.e., the statistical properties of the immediate future are uniquely determined by the present, regardless of the past (see, e.g. [15, 41]). The N -dimensional Markov process can be defined by the Ito stochastic differential equation,

$$dx_i(t) = A_i[x^i(t), t]dt + B_{ij}[x^i(t), t]dW^j(t), \quad (28)$$

$$x^i(0) = x_{i0}, \quad (i, j = 1, \dots, N) \quad (29)$$

or corresponding *Ito stochastic integral equation*

$$x^i(t) = x^i(0) + \int_0^t ds A_i[x^i(s), s] + \int_0^t dW^j(s) B_{ij}[x^i(s), s], \quad (30)$$

in which $x^i(t)$ is the variable of interest, the vector $A_i[x(t), t]$ denotes deterministic *drift*, the matrix $B_{ij}[x(t), t]$ represents continuous stochastic *diffusion fluctuations*, and $W^j(t)$ is an N -variable *Wiener process* (i.e., generalized Brownian motion [15]) and

$$dW^j(t) = W^j(t + dt) - W^j(t).$$

The two Ito equations (29)–(30) are equivalent to the general *Chapman–Kolmogorov probability equation* (see equation (31) below). There are three well known special cases of the Chapman–Kolmogorov equation (see [15]):

1. When both $B_{ij}[x(t), t]$ and $W(t)$ are zero, i.e., in the case of pure deterministic motion, it reduces to the *Liouville equation*

$$\partial_t P(x', t' | x'', t'') = - \sum_i \frac{\partial}{\partial x^i} \{A_i[x(t), t] P(x', t' | x'', t'')\}.$$

2. When only $W(t)$ is zero, it reduces to the *Fokker–Planck equation*

$$\begin{aligned} \partial_t P(x', t' | x'', t'') &= - \sum_i \frac{\partial}{\partial x^i} \{A_i[x(t), t] P(x', t' | x'', t'')\} \\ &+ \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x^i \partial x^j} \{B_{ij}[x(t), t] P(x', t' | x'', t'')\}. \end{aligned}$$

3. When both $A_i[x(t), t]$ and $B_{ij}[x(t), t]$ are zero, i.e., the state-space consists of integers only, it reduces to the *Master equation* of discontinuous jumps

$$\partial_t P(x', t' | x'', t'') = \int dx W(x' | x'', t) P(x', t' | x'', t'') - \int dx W(x'' | x', t) P(x', t' | x'', t'').$$

The *Markov assumption* can now be formulated in terms of the conditional probabilities $P(x^i, t_i)$: if the times t_i increase from right to left, the conditional probability is determined entirely by the knowledge of the most recent condition. Markov process is generated by a set of conditional probabilities whose probability-density $P = P(x', t' | x'', t'')$ evolution obeys the general *Chapman–Kolmogorov integro-differential equation*

$$\begin{aligned} \partial_t P &= - \sum_i \frac{\partial}{\partial x^i} \{A_i[x(t), t] P\} + \frac{1}{2} \sum_{ij} \frac{\partial^2}{\partial x^i \partial x^j} \{B_{ij}[x(t), t] P\} \\ &+ \int dx \{W(x' | x'', t) P - W(x'' | x', t) P\} \end{aligned}$$

including *deterministic drift*, *diffusion fluctuations* and *discontinuous jumps* (given respectively in the first, second and third terms on the r.h.s.). This general Chapman–Kolmogorov integro-differential equation (31), with its conditional probability density evolution, $P = P(x', t' | x'', t'')$, is represented by our SFT-partition function (27).

Furthermore, discretization of the adaptive SFT-partition function (27) gives the standard *partition function*

$$Z = \sum_j e^{-w_j E^j / T}, \quad (31)$$

where E^j is the motion energy eigenvalue (reflecting each possible motivational energetic state), T is the temperature-like environmental control parameter, and the sum runs over all ID energy eigenstates (labelled by the index j). From (31), we can calculate the *transition entropy*, as $S = k_B \ln Z$ (see the next section).

4 Entropy, chaos and phase transitions in the crowd manifold

Recall that nonequilibrium phase transitions [17, 18, 19, 20, 21] are phenomena which bring about qualitative physical changes at the macroscopic level in presence of the same microscopic forces acting among the constituents of a system. In this section we extend the \mathcal{CD} formalism to incorporate both algorithmic and geometrical entropy as well as dynamical chaos [62, 52, 42, 50] between the entropy-growing phase of Mental Preparation and the entropy-conserving phase of Physical Action, together with the associated topological phase transitions.

4.1 Algorithmic entropy

The Boltzmann and Shannon (hence also Gibbs entropy, which is Shannon entropy scaled by $k \ln 2$, where k is the Boltzmann constant) entropy definitions involve the notion of *ensembles*. Membership of microscopic states in ensembles defines the probability density function that underpins the entropy function; the result is that the entropy of a definite and completely known microscopic state is precisely zero. Boltzmann entropy defines the probabilistic model of the system by effectively discarding part of the information about the system, while the Shannon entropy is concerned with measuring the ignorance of the observer – the amount of missing information – about the system.

Zurek proposed a new physical entropy measure that can be applied to individual microscopic system states and does not use the ensemble structure. This is based on the notion of a fixed individually random object provided by Algorithmic Information Theory and Kolmogorov Complexity: put simply, the randomness $K(x)$ of a binary string x is the length in terms of number of bits of the smallest program p on a universal computer that can produce x .

While this is the basic idea, there are some important technical details involved with this definition. The randomness definition uses the prefix complexity $K(\cdot)$ rather than the older Kolmogorov complexity measure $C(\cdot)$: the prefix complexity $K(x|y)$ of x given y is the Kolmogorov complexity $C_{\phi_u}(x|y) = \min \{p \mid x = \phi_u(\langle y, p \rangle)\}$ (with the convention that $C_{\phi_u}(x|y) = \infty$ if there is no such p) that is taken with respect to a reference universal partial recursive function ϕ_u that is a universal prefix function. Then the prefix complexity $K(x)$ of x is just $K(x|\varepsilon)$ where ε is the empty string. A partial recursive prefix function $\phi : M \rightarrow \mathbb{N}$ is a partial recursive function such that if $\phi(p) < \infty$ and $\phi(q) < \infty$ then p is not a proper prefix of q : that is, we restrict the complexity definition to a set of strings (which are descriptions of effective procedures) such that none is a proper prefix of any other. In this way, all effective procedure descriptions are *self-delimiting*: the total length of the description is given within the description itself. A universal prefix function ϕ_u is a prefix function such that $\forall n \in \mathbb{N} \phi_u(\langle y, \langle n, p \rangle \rangle) = \phi_n(\langle y, p \rangle)$, where ϕ_n is numbered n according to some Godel numbering of the partial recursive functions; that is, a universal prefix function is a partial recursive function that simulates any partial recursive function. Here, $\langle x, y \rangle$ stands for a total recursive one-one mapping from $\mathbb{N} \times \mathbb{N}$ into \mathbb{N} , $\langle x_1, x_2, \dots, x_n \rangle = \langle x_1, \langle x_2, \dots, x_n \rangle \rangle$, \mathbb{N} is the set of natural numbers, and $M = \{0, 1\}^*$ is the set of all binary strings.

This notion of entropy circumvents the use of probability to give a concept of entropy that can be applied to a fully specified macroscopic state: the algorithmic randomness of the state is the length of the shortest possible effective description of it. To illustrate, suppose for the moment that the set of microscopic states is countably infinite, with each state identified with some natural number. It is known that the discrete version of the Gibbs entropy (and hence of Shannon's entropy) and the algorithmic entropy are asymptotically consistent under mild assumptions. Consider a system with a countably infinite set of microscopic states X supporting a probability density function $P(\cdot)$ so that $P(x)$ is the probability that the system is in microscopic state $x \in X$. Then the Gibbs entropy is $S_G(P) = -(k \ln 2) \sum_{x \in X} P(x) \log P(x)$ (which is Shannon's information-theoretic entropy $H(P)$ scaled by $k \ln 2$). Supposing that $P(\cdot)$ is recursive, then $S_G(P) = (k \ln 2) \sum_{x \in X} P(x) K(x) + C$, where C_ϕ is a constant depending only on the choice of the reference universal prefix function ϕ . Hence, as a measure of entropy, the function $K(\cdot)$ manifests the same kind of behavior as Shannon's and Gibbs entropy measures.

Zurek's proposal was of a new physical entropy measure that includes contributions from both the randomness of a state and ignorance about it. Assume now that we have determined the macroscopic parameters of the system, and encode this as a string - which can always be converted into an equivalent binary string, which is just a natural number under a standard encoding. It is standard to denote the binary string and its corresponding natural number interchangeably; here let x be the encoded macroscopic parameters. Zurek's definition of *algorithmic entropy* of the macroscopic state is then $K(x) + H_x$, where $H_x = S_B(x)/(k \ln 2)$, where $S_B(x)$ is the Boltzmann entropy of the system constrained by x and k is Boltzmann's constant; the physical version of the algorithmic entropy is therefore defined as $S_A(x) = (k \ln 2)(K(x) + H_x)$. Here H_x represents the level of ignorance about the microscopic state, given the parameter set x ; it can decrease towards

zero as knowledge about the state of the system increases, at which point the algorithmic entropy reduces to the Boltzmann entropy.

4.2 Ricci flow and Perelman entropy–action on the crowd manifold

Recall that the inertial metric crowd flow, $C_t : t \mapsto (M(t), g(t))$ on the crowd $3n$ –manifold (17) is a one-parameter family of homeomorphic Riemannian manifolds (M, g) , evolving by the Ricci flow (25)–(26).

Now, given a smooth scalar function $u : M \rightarrow \mathbb{R}$ on the Riemannian crowd $3n$ –manifold M , its Laplacian operator Δ is locally defined as

$$\Delta u = g^{ij} \nabla_i \nabla_j u,$$

where ∇_i is the covariant derivative (or, Levi–Civita connection). We say that a smooth function $u : M \times [0, T) \rightarrow \mathbb{R}$, where $T \in (0, \infty]$, is a solution to the heat equation on M if

$$\partial_t u = \Delta u. \quad (32)$$

One of the most important properties satisfied by the heat equation is the maximum principle, which says that for any smooth solution to the heat equation, whatever point-wise bounds hold at $t = 0$ also hold for $t > 0$ [6]. This property exhibits the smoothing behavior of the heat diffusion (32) on M .

Closely related to the heat diffusion (32) is the (the Fields medal winning) Perelman entropy–action functional, which is on a $3n$ –manifold M with a Riemannian metric g_{ij} and a (temperature-like) scalar function f given by [64]

$$\mathcal{E} = \int_M (R + |\nabla f|^2) e^{-f} d\mu \quad (33)$$

where R is the scalar Riemann curvature on M , while $d\mu$ is the volume $3n$ –form on M , defined as

$$d\mu = \sqrt{\det(g_{ij})} dx^1 \wedge dx^2 \wedge \dots \wedge dx^{3n}. \quad (34)$$

During the Ricci flow (25)–(26) on the crowd manifold (17), that is, during the inertial metric crowd flow, $C_t : t \mapsto (M(t), g(t))$, the Perelman entropy functional (33) evolves as

$$\partial_t \mathcal{E} = 2 \int |R_{ij} + \nabla_i \nabla_j f|^2 e^{-f} d\mu. \quad (35)$$

Now, the *crowd breathers* are solitonic crowd behaviors, which could be given by localized periodic solutions of some nonlinear soliton PDEs, including the exactly solvable sine–Gordon equation and the focusing nonlinear Schrödinger equation. In particular, the time–dependent crowd inertial metric $g_{ij}(t)$, evolving by the Ricci flow $g(t)$ given by (25)–(26) on the crowd $3n$ –manifold M is the *Ricci crowd breather*, if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g_{ij}(t_1)$ and $g_{ij}(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1, \alpha < 1, \alpha > 1$ correspond to steady, shrinking and expanding crowd breathers, respectively. Trivial crowd breathers, for which the metrics $g_{ij}(t_1)$ and $g_{ij}(t_2)$ on M differ only by diffeomorphism and scaling for each pair of t_1 and t_2 , are the *crowd Ricci solitons*. Thus, if we consider the Ricci flow (25)–(26) as a biodynamical system on

the space of Riemannian metrics modulo diffeomorphism and scaling, then crowd breathers and solitons correspond to periodic orbits and fixed points respectively. At each time the Ricci soliton metric satisfies on M an equation of the form [64]

$$R_{ij} + cg_{ij} + \nabla_i b_j + \nabla_j b_i = 0,$$

where c is a number and b_i is a 1-form; in particular, when $b_i = \frac{1}{2}\nabla_i a$ for some function a on M , we get a gradient Ricci soliton.

Define $\lambda(g_{ij}) = \inf \mathcal{E}(g_{ij}, f)$, where infimum is taken over all smooth f , satisfying

$$\int_M e^{-f} d\mu = 1. \quad (36)$$

$\lambda(g_{ij})$ is the lowest eigenvalue of the operator $-4\Delta + R$. Then the entropy evolution formula (35) implies that $\lambda(g_{ij}(t))$ is non-decreasing in t , and moreover, if $\lambda(t_1) = \lambda(t_2)$, then for $t \in [t_1, t_2]$ we have $R_{ij} + \nabla_i \nabla_j f = 0$ for f which minimizes \mathcal{E} on M [64]. Therefore, a steady breather on M is necessarily a steady soliton.

If we define the conjugate heat operator on M as

$$\square^* = -\partial/\partial t - \Delta + R$$

then we have the conjugate heat equation: $\square^* u = 0$.

The entropy functional (33) is nondecreasing under the coupled Ricci–diffusion flow on M [48]

$$\partial_t g_{ij} = -2R_{ij}, \quad \partial_t u = -\Delta u + \frac{R}{2}u - \frac{|\nabla u|^2}{u}, \quad (37)$$

where the second equation ensures $\int_M u^2 d\mu = 1$, to be preserved by the Ricci flow $g(t)$ on M . If we define $u = e^{-\frac{f}{2}}$, then (37) is equivalent to f –evolution equation on M (the nonlinear backward heat equation),

$$\partial_t f = -\Delta f + |\nabla f|^2 - R,$$

which instead preserves (36). The coupled Ricci–diffusion flow (37) is the most general biodynamic model of the crowd reaction–diffusion processes on M . In a recent study [1] this general model has been implemented for modelling a generic perception–action cycle with applications to robot navigation in the form of a dynamical grid.

Perelman’s functional \mathcal{E} is analogous to negative thermodynamic entropy [64]. Recall that thermodynamic partition function for a generic canonical ensemble at temperature β^{-1} is given by

$$Z = \int e^{-\beta E} d\omega(E), \quad (38)$$

where $\omega(E)$ is a ‘density measure’, which does not depend on β . From it, the average energy is given by $\langle E \rangle = -\partial_\beta \ln Z$, the entropy is $S = \beta \langle E \rangle + \ln Z$, and the fluctuation is $\sigma = \langle (E - \langle E \rangle)^2 \rangle = \partial_{\beta^2} \ln Z$.

If we now fix a closed $3n$ –manifold M with a probability measure m and a metric $g_{ij}(\tau)$ that depends on the temperature τ , then according to equation

$$\partial_\tau g_{ij} = 2(R_{ij} + \nabla_i \nabla_j f),$$

the partition function (38) is given by

$$\ln Z = \int (-f + \frac{n}{2}) dm. \quad (39)$$

From (39) we get (see [64])

$$\begin{aligned} \langle E \rangle &= -\tau^2 \int_M (R + |\nabla f|^2 - \frac{n}{2\tau}) dm, & S &= - \int_M (\tau(R + |\nabla f|^2) + f - n) dm, \\ \sigma &= 2\tau^4 \int_M |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 dm, & \text{where } dm &= u dV, \quad u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}. \end{aligned}$$

From the above formulas, we see that the fluctuation σ is nonnegative; it vanishes only on a gradient shrinking soliton. $\langle E \rangle$ is nonnegative as well, whenever the flow exists for all sufficiently small $\tau > 0$. Furthermore, if the heat function u : (a) tends to a δ -function as $\tau \rightarrow 0$, or (b) is a limit of a sequence of partial heat functions u_i , such that each u_i tends to a δ -function as $\tau \rightarrow \tau_i > 0$, and $\tau_i \rightarrow 0$, then the entropy S is also nonnegative. In case (a), all the quantities $\langle E \rangle, S, \sigma$ tend to zero as $\tau \rightarrow 0$, while in case (b), which may be interesting if $g_{ij}(\tau)$ becomes singular at $\tau = 0$, the entropy S may tend to a positive limit.

4.3 Chaotic inter-phase in crowd dynamics induced by its Riemannian geometry change

Recall that \mathcal{CD} transition map (5) is defined by the chaotic crowd phase-transition amplitude

$$\left\langle \text{PHYS. ACTION} \Big|_{\partial_t S=0} \text{CHAOS} \Big|_{\partial_t S>0} \text{MENTAL PREP.} \right\rangle := \int_M \mathcal{D}[x] e^{iA[x]},$$

where we expect the inter-phase chaotic behavior (see [45]). To show that this chaotic inter-phase is caused by the change in Riemannian geometry of the crowd $3n$ -manifold M , we will first simplify the \mathcal{CD} action functional (18) as

$$A[x] = \frac{1}{2} \int_{t_{ini}}^{t_{fin}} [g_{ij} \dot{x}^i \dot{x}^j - V(x, \dot{x})] dt, \quad (40)$$

with the associated standard Hamiltonian, corresponding to the amalgamate version of (14),

$$H(p, x) = \sum_{i=1}^N \frac{1}{2} p_i^2 + V(x, \dot{x}), \quad (41)$$

where p_i are the $\text{SE}(2)$ -momenta, canonically conjugate to the individual agents' $\text{SE}(2)$ -coordinates x^i , ($i = 1, \dots, 3n$). Biodynamics of systems with action (40) and Hamiltonian (41) are given by the set of *geodesic equations* [41, 44]

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \quad (42)$$

where Γ_{jk}^i are the Christoffel symbols of the affine Levi-Civita connection of the Riemannian \mathcal{CD} manifold M . In this geometrical framework, the instability of the trajectories is the instability of

the geodesics, and it is completely determined by the curvature properties of the \mathcal{CD} manifold M according to the Jacobi equation of geodesic deviation [41, 44]

$$\frac{D^2 J^i}{ds^2} + R^i_{jkm} \frac{dx^j}{ds} J^k \frac{dx^m}{ds} = 0, \quad (43)$$

whose solution J , usually called Jacobi variation field, locally measures the distance between nearby geodesics; D/ds stands for the covariant derivative along a geodesic and R^i_{jkm} are the components of the Riemann curvature tensor of the \mathcal{CD} manifold M .

The relevant part of the Jacobi equation (43) is given by the tangent dynamics equation [8, 5]

$$\ddot{J}^i + R^i_{0k0} J^k = 0, \quad (i, k = 1, \dots, 3n), \quad (44)$$

where the only non-vanishing components of the curvature tensor of the \mathcal{CD} manifold M are

$$R^i_{0k0} = \partial^2 V / \partial x^i \partial x^k. \quad (45)$$

The tangent dynamics equation (44) can be used to define Lyapunov exponents in dynamical systems given by the Riemannian action (40) and Hamiltonian (41), using the formula [7]

$$\lambda_1 = \lim_{t \rightarrow \infty} 1/2t \log(M_{i=1}^N [J_i^2(t) + J_i^2(t)] / M_{i=1}^N [J_i^2(0) + J_i^2(0)]). \quad (46)$$

Lyapunov exponents measure the strength of dynamical chaos in the crowd behavior dynamics. The sum of positive Lyapunov exponents defines the *Kolmogorov–Sinai entropy*.

4.4 Crowd nonequilibrium phase transitions induced by manifold topology change

Now, to relate these results to topological phase transitions within the \mathcal{CD} manifold M given by (17), recall that any two high-dimensional manifolds M_v and $M_{v'}$ have the same topology if they can be continuously and differentiably deformed into one another, that is if they are diffeomorphic. Thus by topology change the ‘loss of diffeomorphicity’ is meant [67]. In this respect, the so-called topological theorem [13] says that non-analyticity is the ‘shadow’ of a more fundamental phenomenon occurring in the system’s configuration manifold (in our case the \mathcal{CD} manifold): a topology change within the family of equipotential hypersurfaces

$$M_v = \{(x^1, \dots, x^{3n}) \in \mathbb{R}^{3n} \mid V(x^1, \dots, x^{3n}) = v\},$$

where V and x^i are the microscopic interaction potential and coordinates respectively. This topological approach to PTs stems from the numerical study of the dynamical counterpart of phase transitions, and precisely from the observation of discontinuous or cuspy patterns displayed by the largest Lyapunov exponent λ_1 at the transition energy [7]. Lyapunov exponents cannot be measured in laboratory experiments, at variance with thermodynamic observables, thus, being genuine dynamical observables they are only be estimated in numerical simulations of the microscopic dynamics. If there are critical points of V in configuration space, that is points $x_c = [\bar{x}_1, \dots, \bar{x}_{3n}]$ such that $\nabla V(x)|_{x=x_c} = 0$, according to the Morse Lemma [32], in the neighborhood of any critical point x_c there always exists a coordinate system $x(t) = [x^1(t), \dots, x^{3n}(t)]$ for which [7]

$$V(x) = V(x_c) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_{3n}^2, \quad (47)$$

where k is the index of the critical point, i.e., the number of negative eigenvalues of the Hessian of the potential energy V . In the neighborhood of a critical point of the \mathcal{CD} -manifold M , equation (47) yields the simplified form of (45), $\partial^2 V / \partial x^i \partial x^j = \pm \delta_{ij}$, giving j unstable directions that contribute to the exponential growth of the norm of the tangent vector J .

This means that the strength of dynamical chaos within the \mathcal{CD} -manifold M , measured by the largest Lyapunov exponent λ_1 given by (46), is affected by the existence of critical points x_c of the potential energy $V(x)$. However, as $V(x)$ is bounded below, it is a good Morse function, with no vanishing eigenvalues of its Hessian matrix. According to Morse theory [32], the existence of critical points of V is associated with topology changes of the hypersurfaces $\{M_v\}_{v \in \mathbb{R}}$. The topology change of the $\{M_v\}_{v \in \mathbb{R}}$ at some v_c is a necessary condition for a phase transition to take place at the corresponding energy value [13]. The topology changes implied here are those described within the framework of Morse theory through ‘attachment of handles’ [32] to the \mathcal{CD} -manifold M .

In our path-integral language this means that suitable topology changes of equipotential sub-manifolds of the \mathcal{CD} -manifold M can entail thermodynamic-like phase transitions [17, 18, 19], according to the general formula:

$$\langle \text{phase out} | \text{phase in} \rangle := \int_{\text{top-ch}} \mathcal{D}[w\Phi] e^{iS[\Phi]}.$$

The statistical behavior of the crowd biodynamics system with the action functional (40) and the Hamiltonian (41) is encompassed, in the canonical ensemble, by its partition function, given by the Hamiltonian path integral [44]

$$Z_{3n} = \int_{\text{top-ch}} \mathcal{D}[p] \mathcal{D}[x] \exp\{i \int_t^{t'} [p_i \dot{x}^i - H(p, x)] d\tau\}, \quad (48)$$

where we have used the shorthand notation

$$\int_{\text{top-ch}} \mathcal{D}[p] \mathcal{D}[x] \equiv \int \prod_{\tau} \frac{dx(\tau) dp(\tau)}{2\pi}.$$

The path integral (48) can be calculated as the partition function [12],

$$\begin{aligned} Z_{3n}(\beta) &= \int \prod_{i=1}^{3n} dp_i dx^i e^{-\beta H(p, x)} = \left(\frac{\pi}{\beta}\right)^{\frac{3n}{2}} \int \prod_{i=1}^{3n} dx^i e^{-\beta V(x)} \\ &= \left(\frac{\pi}{\beta}\right)^{\frac{3n}{2}} \int_0^\infty dv e^{-\beta v} \int_{M_v} \frac{d\sigma}{\|\nabla V\|}, \end{aligned} \quad (49)$$

where the last term is written using the so-called co-area formula [10], and v labels the equipotential hypersurfaces M_v of the \mathcal{CD} manifold M ,

$$M_v = \{(x^1, \dots, x^{3n}) \in \mathbb{R}^{3n} | V(x^1, \dots, x^{3n}) = v\}.$$

Equation (49) shows that the relevant statistical information is contained in the canonical configurational partition function

$$Z_{3n}^C = \int \prod dx^i V(x) e^{-\beta V(x)}.$$

Note that Z_{3n}^C is decomposed, in the last term of (49), into an infinite summation of geometric integrals,

$$\int_{M_v} d\sigma / \|\nabla V\|,$$

defined on the $\{M_v\}_{v \in \mathbb{R}}$. Once the microscopic interaction potential $V(x)$ is given, the configuration space of the system is automatically foliated into the family $\{M_v\}_{v \in \mathbb{R}}$ of these equipotential hypersurfaces. Now, from standard statistical mechanical arguments we know that, at any given value of the inverse temperature β , the larger the number $3n$, the closer to $M_v \equiv M_{u_\beta}$ are the microstates that significantly contribute to the averages, computed through $Z_{3n}(\beta)$, of thermodynamic observables. The hypersurface M_{u_β} is the one associated with

$$u_\beta = (Z_{3n}^C)^{-1} \int \prod dx^i V(x) e^{-\beta V(x)},$$

the average potential energy computed at a given β . Thus, at any β , if $3n$ is very large the effective support of the canonical measure shrinks very close to a single $M_v = M_{u_\beta}$. Hence, the basic origin of a phase transition lies in a suitable topology change of the $\{M_v\}$, occurring at some v_c [12]. This topology change induces the singular behavior of the thermodynamic observables at a phase transition. It is conjectured that the counterpart of a phase transition is a breaking of diffeomorphicity among the surfaces M_v , it is appropriate to choose a diffeomorphism invariant to probe if and how the topology of the M_v changes as a function of v . Fortunately, such a topological invariant exists, the Euler characteristic of the crowd manifold M , defined by [41, 44]

$$\chi(M) = \sum_{k=0}^{3n} (-1)^k b_k(M), \quad (50)$$

where the Betti numbers $b_k(M)$ are diffeomorphism invariants (b_k are the dimensions of the de Rham's cohomology groups $H^k(M; \mathbb{R})$; therefore the b_k are integers). This homological formula can be simplified by the use of the Gauss–Bonnet theorem, that relates $\chi(M)$ with the total Gauss–Kronecker curvature K_G of the \mathcal{CD} -manifold M given by [44, 50]

$$\chi(M) = \int_M K_G d\mu, \quad \text{where } d\mu \text{ is given by (34).}$$

5 Conclusion

Our understanding of crowd dynamics is presently limited in important ways; in particular, the lack of a geometrically *predictive* theory of crowd behavior restricts the ability for authorities to intervene appropriately, or even to recognize when such intervention is needed. This is not merely an idle theoretical investigation: given increasing population sizes and thus increasing opportunity for the formation of large congregations of people, death and injury due to trampling and crushing – even within crowds that have not formed under common malicious intent – is a growing concern among police, military and emergency services. This paper represents a contribution towards the understanding of crowd behavior for the purpose of better informing decision-makers about the dangers and likely consequences of different intervention strategies in particular circumstances.

In this paper, we have proposed an entropic geometrical model of crowd dynamics, with dissipative kinematics, that operates across macro-, micro- and meso-levels. This proposition is motivated by the need to explain the dynamics of crowds across these levels simultaneously. We contend that only by doing this can we expect to adequately characterize the geometrical properties of crowds with respect to regimes of behavior and the changes of state that mark the boundaries between such regimes.

In pursuing this idea, we have set aside traditional assumptions with respect to the separation of mind and body. Furthermore, we have attempted to transcend the long-running debate between contagion and convergence theories of crowd behavior with our multi-layered approach: rather than representing a reduction of the whole into parts or the emergence of the whole from the parts, our approach is build on the supposition that the direction of logical implication can and does flow in both directions simultaneously. We refer to this third alternative, which effectively unifies the other two, as *behavioral composition*.

The most natural statistical descriptor is crowd entropy, which satisfies the extended second thermodynamics law applicable to open systems comprised of many components. Similarities between the configuration manifolds of individual (micro-level) and crowds (macro-level) motivate our claim that goal-directed movement operates under entropy conservation, while natural crowd dynamics operates under monotonically increasing entropy functions. Of particular interest is what happens between these distinct topological phases: the phase transition is marked by chaotic movement.

We contend that this approach provides a basis on which one can build a geometrically predictive model of crowd behavior dynamics – over and above the existing approaches, which are largely explanatory. The current paper develops an entropy formulation of crowd dynamics as a three-step process involving individual and collective behavior dynamics, and - crucially - non-equilibrium phase transitions whereby the forces operating at the microscopic level result in geometrical change at the macroscopic level. We have incorporated both geometrical and algorithmic notions of entropy as well as chaos in studying the topological phase transition between the entropy conservation of physical action and the entropy increase during internal, action preparation stages of behavior. Given these formulations, future research can focus on: (i) crowd simulations in 3D graphics environments, (ii) motivated cognition underpinning crowd dynamics and (iii) mechanisms of abrupt change in crowd behavior, such as crowd turbulence and flashpoints.

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6 Appendix

General nonlinear stochastic dynamics, developed in a framework of Feynman path integrals, have recently [46] been applied to Lewinian field-theoretic psychodynamics [58], resulting in the development of a new concept of Life-Space Foam (LSF) as a natural medium for motivational and cognitive psychodynamics. According to the LSF-formalism, the classic Lewinian life space can be macroscopically represented as a smooth manifold with steady force-fields and behavioral paths,

while at the microscopic level it is more realistically represented as a collection of wildly fluctuating force-fields, (loco)motion paths and local geometries (and topologies with holes).

A set of least-action principles is used to model the smoothness of global, macro-level LSF paths, fields and geometry, according to the following prescription. The action $S[\Phi]$, with dimensions of $Energy \times Time = Effort$ and depending on macroscopic paths, fields and geometries (commonly denoted by an abstract field symbol Φ^i) is defined as a temporal integral from the initial time instant t_{ini} to the final time instant t_{fin} ,

$$S[\Phi] = \int_{t_{ini}}^{t_{fin}} \mathcal{L}[\Phi] dt, \quad (51)$$

with Lagrangian density given by

$$\mathcal{L}[\Phi] = \int d^n x \mathcal{L}(\Phi_i, \partial_{x^j} \Phi^i),$$

where the integral is taken over all n coordinates $x^j = x^j(t)$ of the LSF, and $\partial_{x^j} \Phi^i$ are time and space partial derivatives of the Φ^i -variables over coordinates. The standard least action principle

$$\delta S[\Phi] = 0, \quad (52)$$

gives, in the form of the so-called Euler-Lagrangian equations, a shortest (loco)motion path, an extreme force-field, and a life-space geometry of minimal curvature (and without holes). In this way, we have obtained macro-objects in the global LSF: a single path described by Newtonian-like equation of motion, a single force-field described by Maxwellian-like field equations, and a single obstacle-free Riemannian geometry (with global topology without holes).

To model the corresponding local, micro-level LSF structures of rapidly fluctuating MD & CD, an adaptive path integral is formulated, defining a multi-phase and multi-path (multi-field and multi-geometry) transition amplitude from the motivational state of *Intention* to the cognitive state of *Action*,

$$\langle Action | Intention \rangle_{total} := \int \mathcal{D}[w\Phi] e^{iS[\Phi]}, \quad (53)$$

where the Lebesgue integration is performed over all continuous $\Phi_{con}^i = paths + fields + geometries$, while summation is performed over all discrete processes and regional topologies Φ_{dis}^j . The symbolic differential $\mathcal{D}[w\Phi]$ in the general path integral (20) represents an adaptive path measure, defined as a weighted product

$$\mathcal{D}[w\Phi] = \lim_{N \rightarrow \infty} \prod_{s=1}^N w_s d\Phi_s^i, \quad (i = 1, \dots, n = con + dis). \quad (54)$$

The adaptive path integral (53)–(54) represents an ∞ -dimensional neural network, with weights w updating by the general rule [49]

$$new\ value(t+1) = old\ value(t) + innovation(t).$$

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